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Transformation Systems for Incidence Structures

GLORIA RINALDI AND LUIGI ANTONIO ROSATI[†]

By taking into account the transformation technique of Quattrocchi and Rosati, we study how to generate transformation systems for an incidence structure by starting from some given generating blocks and by a suitable permutation of the points. Furthermore, the method is applied to obtain the finite André planes by means of a minimal set of generating lines of Desarguesian planes.

1. INTRODUCTION

Let t and k be cardinal numbers, t finite. A (t, k)-structure is an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ such that each block contains k points, any t distinct points $p_1, \ldots, p_t$ are incident with a unique block, denoted by $\langle p_1, \ldots, p_t \rangle$. In [3], Quattrocchi and Rosati introduced a method to transform a (t, k)-structure into another one, with the same parameters t and k, but not necessarily isomorphic to the previous one. More precisely, let $\mathcal{F}$ be a family of blocks: $\mathcal{F} \subset \mathcal{B}$ and let $f \in \mathrm{Sym}\,\mathcal{P}$; the pair $(\mathcal{F}, f)$ is called a *transformation system* for $\mathcal{S}$ if the following condition $(^*)$ holds:

$$(^*) \qquad \langle p_1, \ldots, p_t \rangle \in \mathcal{F} \iff \langle f(p_1), \ldots, f(p_t) \rangle \in \mathcal{F}.$$

Given a transformation system for $\mathcal{S}$, a new incidence structure $\mathcal{S}' = (\mathcal{P}, \mathcal{B}, \mathcal{J})$ which preserves the point and block sets of $\mathcal{S}$ can be defined by changing the incidence relation as follows:

(1) If $B \in \mathcal{F}$ then $pJB \iff f(p)IB$;
(2) If $B \in \mathcal{B} - \mathcal{F}$ then $pJB \iff pIB$.

The new incidence structure $\mathcal{S}'$ is again a (t, k)-structure, [3]. Observe that if $\mathcal{S}$ is finite, that is $\mathcal{P}$ is finite, $(^*)$ is equivalent to

$$(^*)' \qquad \langle p_1, \ldots, p_t \rangle \in \mathcal{F} \Longrightarrow \langle f(p_1), \ldots, f(p_t) \rangle \in \mathcal{F}.$$

Furthermore, if $\mathcal{S}$ is a finite plane, then $\mathcal{S}'$ is a plane of the same order.

Several applications of this transformation technique were found, see [1, 3–6]. In these papers, the problem of constructing a transformation system for a given (t, k)-structure was not considered, but mainly properties and applications of the transformation technique were studied under the assumption that the transformation system was known.

Our paper now focuses the attention on the problem of constructing transformation systems. Precisely, if $\mathcal{S}$ is a (t, k)-structure, f is a permutation of its point-set and $B_1, \ldots, B_r$ are some given blocks, Propositions 1 and 2 of Section 2 give a method to construct a transformation system $(\mathcal{F}, f)$ such that $B_1, \ldots, B_r$ are elements of $\mathcal{F}$.

To clarify better the aim of our paper, consider the following example. Suppose S is a finite Desarguesian plane, let r and r' be two distinct lines of $\mathcal{S}$ and let $A_1, A_2, A_3, B_1, B_2, B_3$

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be six distinct points with $A_i I r$ and $B_i I r'$, $i \in \{1, 2, 3\}$. Denote by C_i the intersection point of the lines $\langle A_i, B_{i+1}\rangle$, $\langle A_{i+1}, B_i\rangle$, $i \in \{1, 2, 3\}$, ($i + 1$ is taken mod 3) and denote by s the line containing C_1, C_2, C_3. Consider a permutation f on the point-set which fixes the points A_i and B_i for each i, and such that, for a fixed $\bar{i} \in \{1, 2, 3\}$, $f(C_{\bar{i}})$ is not I-incident to s and f fixes the points C_i different from $C_{\bar{i}}$. Then, by starting with the line s and the permutation f, generate a transformation system for the plane as shown in Proposition 1. It is easy to show that if the constructed family $\mathcal{F}$ does not contain the lines $\langle A_{\bar{i}}, B_{\bar{i}+1}\rangle$, $\langle A_{\bar{i}+1}, B_{\bar{i}}\rangle$, then the transformed plane is not Desarguesian. So, if we are able to choose a 'good' permutation and if we have a method to generate a transformation system, we can construct a plane not isomorphic to the one we started with. For example, this is the case when we consider $AG(2, 25)$, $A_1 = (0, 1 + i)$, $A_2 = (3, 1 + 3i)$, $A_3 = (2, 1 + 4i)$, $B_1 = (3, 3)$, $B_2 = (1, 4 + i)$, $B_3 = (0, 2 + 4i)$ with $i^2 + i + 1 = 0$, $f : (x, y) \longrightarrow (x^5, y)$ and we apply Proposition 1.

Some examples of constructions of transformation systems will be given in Section 3. In particular, we will see how for each André plane A of order q it is possible to find a minimal set of lines of $AG(2, q)$ and some automorphisms of $GF(q)$ which generate a transformation system producing A from $AG(2, q)$. In Remark 2, we will also observe how it is possible to construct translation, but not André planes.

2. Generation of Transformation Systems

In this section, $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ will denote a finite (t, k)-structure with point-set $\mathcal{P}$ and block-set $\mathcal{B}$.

PROPOSITION 1. *Let f be a permutation of the point-set and let $B_1, \dots, B_r$ be r given distinct blocks, $r \in N - \{0\}$. Set $\mathcal{F}_0(B_1, \dots, B_r) = \{B_1, \dots, B_r\}$ and, for each $n \in N$, define:*

$$\mathcal{F}_{n+1}(B_1, \dots, B_r) = \{\langle f(p_1), \dots, f(p_t)\rangle | p_i \neq p_j, \qquad \langle p_1, \dots, p_t\rangle \in \mathcal{F}_n(B_1, \dots, B_r)\}.$$

Let $\mathcal{F}_f(B_1, \dots, B_r) = \cup \mathcal{F}_n(B_1, \dots, B_r)$, then $(\mathcal{F}_f(B_1, \dots, B_r), f)$ is a transformation system for $\mathcal{S}$.

PROOF. Condition (*)' holds: let $\langle p_1, \dots, p_t\rangle \in \mathcal{F}_f(B_1, \dots, B_r)$, then there exists $n \in N$ such that $\langle p_1, \dots, p_t\rangle \in \mathcal{F}_n(B_1, \dots, B_r)$, therefore the block $\langle f(p_1), \dots, f(p_t)\rangle$ is an element of $\mathcal{F}_{n+1}(B_1, \dots, B_r)$ and $\mathcal{F}_{n+1}(B_1, \dots, B_r) \subset \mathcal{F}_f(B_1, \dots, B_r)$. □

PROPOSITION 2. *With the notation of Proposition 1, the following hold:*

(1) *$\mathcal{F}_f(B_1, \dots, B_r) = \mathcal{F}_f(B_1) \cup \dots \cup \mathcal{F}_f(B_r)$;*
(2) *For any two blocks B_i, B_j, either $\mathcal{F}_f(B_i) = \mathcal{F}_f(B_j)$ or $\mathcal{F}_f(B_i) \cap \mathcal{F}_f(B_j) = \emptyset$.*

PROOF. To prove (1), we first show that, for every $n \in N$, we have: $\mathcal{F}_n(B_1, \dots, B_r) = \mathcal{F}_n(B_1) \cup \dots \cup \mathcal{F}_n(B_r)$. Proceed by induction on n: obviously $\mathcal{F}_0(B_1, \dots, B_r) = \{B_1, \dots, B_r\} = \mathcal{F}_0(B_1) \cup \dots \cup \mathcal{F}_0(B_r)$. Now suppose $\mathcal{F}_i(B_1, \dots, B_r) = \mathcal{F}_i(B_1) \cup \dots \cup \mathcal{F}_i(B_r)$. Let $B \in \mathcal{F}_{i+1}(B_1, \dots, B_r)$, then there exists $\langle p_1, \dots, p_t\rangle \in \mathcal{F}_i(B_1, \dots, B_r)$ and $B = \langle f(p_1), \dots, f(p_t)\rangle$. By induction, $\langle p_1, \dots, p_t\rangle \in \mathcal{F}_i(B_j)$ for some $j \in \{1, \dots, r\}$. Therefore, B is a block of $F_{i+1}(B_j)$. This implies $\mathcal{F}_{i+1}(B_1, \dots, B_r) \subset \mathcal{F}_{i+1}(B_1) \cup \dots \mathcal{F}_{i+1}(B_r)$. Similarly, $\mathcal{F}_{i+1}(B_1) \cup \dots \cup \mathcal{F}_{i+1}(B_r) \subset \mathcal{F}_{i+1}(B_1, \dots, B_r)$.

To prove (2), we show that $B \in \mathcal{F}_f(B_i)$ implies $\mathcal{F}_f(B) = \mathcal{F}_f(B_i)$; so, whenever $\mathcal{F}_f(B_i) \cap \mathcal{F}_f(B_j) \neq \emptyset$, by taking $B \in \mathcal{F}_f(B_i) \cap \mathcal{F}_f(B_j)$ we have $\mathcal{F}_f(B_i) = \mathcal{F}_f(B) =$

$\mathcal{F}_f(B_j)$. Therefore, let $B \in \mathcal{F}_f(B_i)$ and define $\tau(B) = n$ whenever $B \in \mathcal{F}_n(B_i)$ and $B \notin \mathcal{F}_{n-1}(B_i)$. By induction on n, we prove that $B \in \mathcal{F}_f(B_i)$ implies $\mathcal{F}_f(B) = \mathcal{F}_f(B_i)$. Obviously, $\tau(B) = 0$ means $B = B_i$, that is $\mathcal{F}_f(B) = \mathcal{F}_f(B_i)$. Suppose $\mathcal{F}_f(B) = \mathcal{F}_f(B_i)$ whenever $\tau(B) \leq h$, $h \in N$, and let $\tau(B) = h + 1$. Then $B \in \mathcal{F}_{h+1}(B_i)$, that is $B = \langle f(p_1), \ldots, f(p_t)\rangle$, $\langle p_1, \ldots, p_t\rangle \in \mathcal{F}_h(B_i)$ and $\tau(\langle p_1, \ldots, p_t\rangle) \leq h$. By induction, $\mathcal{F}_f(\langle p_1, \ldots, p_t\rangle) = \mathcal{F}_f(B)$.

We want to prove that $\mathcal{F}_f(\langle p_1, \ldots, p_t\rangle) = \mathcal{F}_f(B)$. Recall that $B = \langle f(p_1), \ldots, f(p_t)\rangle$. Therefore, $B \in \mathcal{F}_f(\langle p_1, \ldots, p_t\rangle)$ and $\mathcal{F}_f(B) \subset \mathcal{F}_f(\langle p_1, \ldots, p_t\rangle)$. For the second inclusion observe that $(\mathcal{F}_f(B), f)$ is a transformation system for $\mathcal{S}$, therefore $B = \langle f(p_1), \ldots, f(p_t)\rangle \in \mathcal{F}_f(B)$ implies $\langle p_1, \ldots, p_t\rangle$ in $\mathcal{F}_f(B)$, that is $\mathcal{F}_f(\langle p_1, \ldots, p_t\rangle) \subset \mathcal{F}_f(B)$. □

REMARK 1. If $\mathcal{S} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ is a (t, k)-structure, $(\mathcal{F}, f)$ is a transformation system for $\mathcal{S}$ and $\{B_1, \ldots, B_r\} \subset \mathcal{F}$, then $(\mathcal{F}_f(B_1, \ldots, B_r), f)$ is a transformation system for $\mathcal{S}$ with $\mathcal{F}_f(B_1, \ldots, B_r) \subset \mathcal{F}$. So we can make the following definition:

DEFINITION 1. The blocks $B_1, \ldots, B_r$ are said to generate the transformation system $(\mathcal{F}, f)$ whenever $\mathcal{F}_f(B_1, \ldots, B_r) = \mathcal{F}$. The set $\{B_1, \ldots, B_r\}$ is said to be a minimal set of generators for $\mathcal{F}$ whenever $\mathcal{F}_f(B_1, \ldots, B_r) = \mathcal{F}$ and $\mathcal{F}_f(B_{i_1}, \ldots, B_{i_s}) \neq \mathcal{F}_f(B_1, \ldots, B_r)$ for each proper subset $\{B_{i_1}, \ldots, B_{i_s}\} \subset \{B_1, \ldots, B_r\}$.

3. THE FINITE ANDRÉ PLANES

Let F be a finite field and σ an automorphism of it. If H is the subfield of F fixed by σ and if H is of order q, then $F = GF(q^h)$ where h is the exponent of σ. In the affine Desarguesian plane $AG(2, q^h)$ denote by $r_{m,k}$ the lines of equation $y = mx + k$, $m \in F^* = F - \{0\}$, $k \in F$, then:

PROPOSITION 3. *Given a line $r_{\bar{m},\bar{k}}$ of $AG(2, q^h)$, let $f : (x, y) \longrightarrow (\sigma(x), y)$ be a permutation of the point-set of $AG(2, q^h)$. Let $A = \{x^{q-1} | x \in F^*\}$; then $\mathcal{F}_f(r_{\bar{m},\bar{k}}) = \{r_{m,k} | m \in A\bar{m}, k \in F\}$.*

PROOF. Let $F = GF(p^n)$, p prime, $\sigma : x \longrightarrow x^{p^j}$ and $s = GCD(n, j)$. Then $q = p^s$ and $n = hs$. Let τ be the automorphism of F with $\tau : x \longrightarrow x^q$ and let Γ be the group generated by τ. Hence, $\sigma = \tau^t$ with $j = ts$ and $\langle\sigma\rangle = \langle\tau\rangle = \Gamma$. Finally, $\{\sigma(x)x^{-1} | x \in F^*\} = A$. Set $T = \{r_{m,k} | m \in A\bar{m}, k \in F\}$.

First we prove that $\mathcal{F}_f(r_{\bar{m},\bar{k}}) \subset T$. In fact, if we take a line $r_{m,k}$, $m = a\bar{m}$, $a \in A$ and $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ are two distinct points on it, the equation of the line containing $f(P_1)$ and $f(P_2)$ is $y = a\bar{m}(x_1 - x_2)[\sigma(x_1 - x_2)]^{-1}x + l$, $a \in A$ and $l \in F$. As A is a group, $\langle f(P_1), f(P_2)\rangle$ is a line of T. To construct $\mathcal{F}_f(r_{\bar{m},\bar{k}})$ we start from the line $r_{\bar{m},\bar{k}}$. As $\bar{m} = 1\bar{m}$ and $1 \in A$, the inclusion follows.

Let $z \in F$ and $P_1 = (z, \bar{m}z + \bar{k})$, $P_2 = (1 + z, \bar{m}(1 + z) + \bar{k})$ be two distinct points of $r_{\bar{m},\bar{k}}$. Thus, the line $\langle f(P_1), f(P_2)\rangle$ has equation: $y = \bar{m}x + \bar{m}(z - \sigma(z)) + \bar{k}$. Observe also that $z_1 - \sigma(z_1) = z_2 - \sigma(z_2)$ implies $z_1 - z_2 \in GF(q)$. Then, as z varies in F, we construct q^{h-1} distinct lines of $\mathcal{F}_f(r_{\bar{m},\bar{k}})$. Next, note that $|A|\rangle q^{h-1}$, therefore the set $\{1 - a | a \in A\}$ contains more than q^{h-1} elements; so it is possible to find an element $b \in F^*$ such that $1 - b^{1-q} \notin \{z - \sigma(z) | z \in F\}$. Choose $t \in H$, $r \in F$ with $t - r = b$ and let $Q_1 = (t, y_1) Q_2 = (r, y_2)$ be two points of a line $y = \bar{m}x + \bar{m}(z - \sigma(z)) + \bar{k}$. Thus, the line $\langle f(Q_1), f(Q_2)\rangle$ is a line of $\mathcal{F}_f(r_{\bar{m},\bar{k}})$ with equation: $y = \bar{m}b^{1-q}x + \bar{m}t(1 - b^{1-q}) + \bar{m}(z - \sigma(z)) + \bar{k}$. Recall that F is a vector space over H of dimension h and $\{z - \sigma(z) | z \in F\}$ is a subspace of F

of dimension $h-1$. Furthermore, $1-b^{q-1}$ generates a complementary subspace. We can conclude that $\{t(1-b^{1-q})+(z-\sigma(z)) | t \in H, z \in F\} = F$. Hence, we have constructed all the lines: $y = \bar{m}b^{1-q}x + k$, $k \in F$ which are contained in $\mathcal{F}_f(r_{\bar{m},\bar{k}})$. Finally, let $a \in A$, $a = d^{1-q}$, $d \in F^*$, and let $T_1 = (0, \bar{k})T_2 = (d, \bar{m}d + \bar{k})$ be two distinct points of $r_{\bar{m},\bar{k}}$. Then $\langle f(T_1), f(T_2)\rangle$ is a line of $\mathcal{F}_f(r_{\bar{m},\bar{k}})$ with equation $y = \bar{m}ax + \bar{k}$. Proceeding as above, we construct all the lines: $y = \bar{m}ab^{1-q}x + k$, $k \in F$. As $A = \{ab^{1-q} | a \in A\}$, it is $T \subset \mathcal{F}_f(r_{\bar{m},\bar{k}})$. Thus, $T = \mathcal{F}_f(r_{\bar{m},\bar{k}})$. □

To see which plane is obtained by $AG(2, q^h)$ with the transformation system $(\mathcal{F}_f(r_{\bar{m},\bar{k}}), f)$, we recall the construction of the finite André planes.

Let F be a finite field and let σ be an automorphism of it. Let $\Gamma = \langle\sigma\rangle$ be the subgroup of $Aut F$ generated by σ and let H be the subfield of F fixed by σ (so by all the elements of Γ). Then if $|H| = q$, we have $F = GF(q^h)$ with $h = |\Gamma|$ and $\Gamma = \langle\tau\rangle$ with $\tau : x \longrightarrow x^q$. Let $A = \{x^{q-1} | x \in F^*\}$.

Let $\mu : F^* \longrightarrow H^*$ be the norm function $\left(\mu(x) = \prod_{\sigma_i \in \Gamma} \sigma_i(x) = x^{1+q+\cdots+q^{h-1}}\right)$.

Let $\phi : H^* \longrightarrow \Gamma$ be a map with the only request that $\phi(1) = id$.

Set $\alpha_m = \phi\mu(m)$. An André plane is obtained by taking the points of $AG(2, q^h)$ and the lines of equation:

$$y = m\alpha_m(x) + k, m, k \in F;$$
$$x = k, k \in F.$$

PROPOSITION 4. *With the notation of Proposition 3, the affine plane obtained by transforming $AG(2, q^h)$ by the transformation system $(\mathcal{F}_f(r_{\bar{m},\bar{k}}), f)$ is an André plane.*

PROOF. Recall the previous description of the André planes and observe that A is a subgroup of the multiplicative group F^* and $\mu(a) = 1$ for each $a \in A$. Therefore, $\alpha_{m_1} = \alpha_{m_2}$ whenever m_1 and m_2 are elements of the same coset. If $\bar{m} \notin A$, then the plane obtained by transforming $AG(2, q^h)$ by the transformation system $(\mathcal{F}_f(r_{\bar{m},\bar{k}}), f)$ is an André plane obtained with $\Gamma = \langle\sigma\rangle$; $\phi\mu(\bar{m}) = \sigma$ and $\phi\mu(x) = id$ whenever $x \notin A\bar{m}$. On the other hand, if $\bar{m} \in A$, let: $\phi\mu(x) = id$ for every $x \in A$ and $\phi\mu(x) = \sigma^{-1}$ for every $x \notin A$. □

REMARK 2. Recall the notation of Propositions 3 and 4 and note that if we fix l distinct lines of $AG(2, q^h)$: $r_1, \ldots, r_l$, then $(\mathcal{F}_f(r_1, \ldots, r_l), f)$ transforms $AG(2, q^h)$ into an André plane of the same order. In fact, we may always suppose $\{r_1, \ldots, r_l\}$ is a minimal set, namely $\mathcal{F}(r_i) \cap \mathcal{F}(r_j) = \emptyset$ whenever $i \neq j$; for each i, suppose r_i has the equation: $y = \bar{m}_i x + \bar{k}_i$. Then the transformed plane contains all the lines: $y = \bar{m}_i m\sigma(x) + k$, $m \in A$, $k \in F$, $i \in \{1, \ldots, l\}$. If for each $i \in \{1, \ldots, l\}$, the coset $A\bar{m}_i$ is different from A, then the transformed plane is an André plane obtained with $\Gamma = \langle\sigma\rangle$ and with $\phi\mu(\bar{m}_i) = \sigma$ for each $i \in \{1, \ldots, l\}$. On the other hand, if $\bar{m}_j \in A$ for some j, then we have an André plane obtained with $\Gamma = \langle\sigma\rangle$, $\phi\mu(x) = id$ for every $x \in A\bar{m}_1 \cup \cdots \cup A\bar{m}_l$ and $\phi\mu(x) = \sigma^{-1}$ for every $x \in F^* - (A\bar{m}_1 \cup \cdots \cup A\bar{m}_l)$.

If we choose two (or more) different automorphisms of F, we may also construct planes which are not André planes. For example, fix $\sigma_1, \sigma_2 \in Aut F$ and two lines r_{m_1,k_1}, r_{m_2,k_2} such that $Am_1 \cap Bm_2 = \emptyset$ with $A = \{\sigma_1(x)x^{-1} | x \in F^*\}$ and $B = \{\sigma_2(x)x^{-1} | x \in F^*\}$; let $f : (x, y) \longrightarrow (\sigma_1(x), y)$; $g : (x, y) \longrightarrow (\sigma_2(x), y)$. Then $(\mathcal{F}_f(r_{m_1,k_1}), f)$ transforms $AG(2, q^h)$ into an André plane: call it $\bar{\pi}$. $(\mathcal{F}_g(r_{m_2,k_2}), g)$ is a transformation system for $\bar{\pi}$. By transforming $\bar{\pi}$, we obtain the plane $\bar{\pi}'$ whose lines have equations:

$$\begin{aligned}
&y = am_1\sigma_1(x) + k, \qquad a \in A, k \in F;\\
&y = bm_2\sigma_2(x) + k, \qquad b \in B, k \in F;\\
&y = mx + k, \qquad m \in F - (Am_1 \cup Bm_2);\\
&x = k, \qquad k \in F.
\end{aligned}$$

In general, this is not an André plane. For example, take $F = GF(p^6)$, $p \neq 2$; let ϵ be a primitive element of $F, \sigma : x \longrightarrow x^{p^2}$, $\tau : x \longrightarrow x^{p^3}$, $m_1 = \epsilon$ and $m_2 = \epsilon^2$. In order to obtain an André plane, σ and τ should belong to a proper subgroup Γ of $AutF$. If so, then $\Gamma = \langle\alpha\rangle$, $\alpha : x \longrightarrow x^p$ and $|\{x^{p-1}|x \in GF(p^6)^*\}|$ divides $|Am_1|$. This is a contradiction, as $|\{x^{p-1}|x \in GF(p^6)^*\}| = p^5 + p^4 + p^3 + p^2 + p + 1$ and $|Am_1| = |A| = p^4 + p^2 + 1$.

Obviously $\bar{\pi}'$ is obtained from $AG(2, q^h)$ by a substitution of nets in the sense of Ostrom [2].

REMARK 3. In [4], it was already proved that a finite André plane can be obtained by transforming the Desarguesian plane of the same order. More precisely, by taking into account the description of the finite André planes given at the beginning of this section, let $\{A_1, A_2, \ldots, A_{q-1}\}$ be the cosets of A in F^* and observe that $\phi\mu$ has the same action on the elements of a same coset of A. Therefore, let $A_1 = A$ and $A_r = A\bar{m}_r$, $r = 2, \ldots, q-1$; the lines of the plane have equations:

$$\begin{aligned}
&y = mx + k, m \in A, k \in F;\\
&y = m\bar{m}_r\sigma_r(x) + k, m \in A, k \in F, \sigma_r \in \Gamma = \langle\sigma\rangle, \text{ as } r \text{ varies in } \{2, \ldots, q-1\}.
\end{aligned}$$

For each $r \in \{2, \ldots, q-1\}$, set $\mathcal{F}_r = \{r_{m\bar{m}_r,k}|m \in A, k \in F\}$, $f_r : (x, y) \longrightarrow (\sigma_r(x), y)$. Thus $(\mathcal{F}_r, f_r)$ is a transformation system for $AG(2, q^h)$ and a sequence of transformations with transformation systems $(\mathcal{F}_2, f_2), \ldots, (\mathcal{F}_{q-1}, f_{q-1})$ allows us to construct the André plane.

We now describe how it is possible to obtain a minimal set of generators for each $(\mathcal{F}_r, f_r)$.

If σ_r is a generator of Γ, then $\{\sigma_r(x)x^{-1}|x \in F^*\} = A$ and the set $\{r_{\bar{m}_r,\bar{k}}\}$, for any fixed $\bar{k} \in F$, is a minimal set of generators for $(\mathcal{F}_r, f_r)$.

Now suppose $\langle\sigma_r\rangle \neq \langle\sigma\rangle$. Hence, $\sigma_r = \tau^t$ with $\tau : x \longrightarrow x^q$ and t and h are not coprime. Let $l \neq 1$ be their greatest common divisor. The set $\mathcal{F}_f(r_{\bar{m}_r,\bar{k}})$, for any $\bar{k} \in F$, is the set $\{r_{\bar{m}_r m,k}|m \in \bar{A}, k \in F\}$ with $\bar{A} = \{\sigma_r(x)x^{-1}|x \in F^*\}$ and it is different from $\mathcal{F}_r$. Notice that $\bar{A}$ is a subgroup of A. Denote by $\bar{A} = \bar{A}a_1, \ldots, \bar{A}a_u$ the cosets of $\bar{A}$ in A and let r_t be the line of equation $y = \bar{m}_r a_t x + k_t$, $a_t \in \{a_1, \ldots, a_u\}$, $k_t \in F$. Therefore: $\mathcal{F}_f(r_1, \ldots, r_u) = \mathcal{F}_f(r_1) \cup \cdots \cup \mathcal{F}_f(r_u)$, $\mathcal{F}_f(r_t) = \{r_{m\bar{m}_r a_t,k}|m \in \bar{A}, k \in F\}$, $\mathcal{F}_f(r_1, \ldots, r_u) = \mathcal{F}_r$. We conclude that $\{r_1, \ldots, r_u\}$ is a minimal set of generators for $(\mathcal{F}_r, f_r)$. Furthermore, as $GF(q^l)$ is the subfield containing all elements fixed by σ_r, we have $|\bar{A}| = \frac{q^h-1}{q^l-1}$. Then $u = \frac{q^l-1}{q-1}$.

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G. RINALDI
Dipartimento di Matematica Univ. Modena,
Via Campi 213/B,
41100 Modena, Italy

L. A. ROSATI
Dipartimento di Matematica Univ Firenze,
Viale Morgagni 67/A,
50134 Firenze, Italy